LECTURE 7

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FOR TODAY

- Primitives
- Generic Timing
- First Price Auction in detail
- Second Price Auction in detail
- Revenue Equivalence

AUCTION: PRIMITIVES 1

- 1 seller (called the auctioneer)
- 1 indivisible item (can't chop it up into pieces)
- $n \geq 1$ buyers
- Seller and Buyers WILLINGNLY participate
- Outside options:

i. if a buyer DOES NOT participate, he gets a payoff of 0

ii. If seller DOES NOT participate, everyone gets a payoff of 0

AUCTION PRIMITIVES 2

- Each buyer i has a valuation for the good
- A valuation is a number $\theta_i \in [0,1] \equiv \Theta$
- It is distributed given a CDF $F(.)$ over valuations
- Valuations over draws are drawn independently across buyers
- The classic theory often allows for each buyer's type $\theta_i \sim F_i(.)$ (a feature I always disregard)
- Remember that one ASSUMES that distributional assumptions are common knowledge (when this is not true, one enters an area of economics called "Robust Mechanism Design")

ROBUST MECHANISM DESIGN

- Main question: What if the seller does not know the true distribution F?
- This literature is led by Dirk Bergemann, Stephen Morris, Santoru Takahashi, Eric Markin, Harry Pei, Bruno Strulovicci, among others.
- Main findings: Monopolies (seller concedes higher rents to consumers, on average); on the whole , viable institutions are often simpler.

WORKING EXAMPLES (OF TYPE)

i. Example 1: finitely many types: $\theta_i \in \Theta = \{0, ½, 1\}, \Pr(\theta_i = 0) = \Pr\left(\theta_i = \frac{1}{2}\right) = \Pr(\theta_i = 1) = \frac{1}{3}$

ii. Example 2: $\forall \theta_i \in \Theta = [0,1], F(\theta_i) = \theta_i$

- I will focus on auctions with two buyers and 1 seller for illustration
- Note that assignments will use these workable examples.

TIMING OF AUCTIONS CONSIDERED

- Timing goes as follows:

1) Nature draws valuations $(\theta_i)_{i=1}^{n}$
2) Nature (privately) informs each buyer i of their valuation θ_i,
3) Seller and Buyers decide whether to enter an auction and follow its procedure
4) Buyers participating in the auction submit bids, other buyers just go home
5) Following a specific auction procedure, seller either gives the good to a buyer i or keeps the good
6) Payments are made following auction rules among willing participants

BUT WHAT DOES A PARTICULAR AUCTION PROCEDURE ENTAILS?

- Previous slide was intentionally vague describing the actual trading procedure
- For this class, I assume a fixed set of actions and allow only the way payments are decided as well as who gets the good to be decided
- Fine point: Economists call who gets the good, the “allocation”
- An allocation can state that a given buyer gets the good or that the seller keeps the good

IMPLICIT ASSUMPTIONS

i. The seller follows through implementing the mechanism he announced

ii. He does not re-offer the good (see Liu, Mierendorff, Shi, and Zhong 2019 , Skreta 2006, Doval and Skreta 2021, and Ramos-Mercado 2021)

iii. It is common knowledge what procedure was chosen by the seller.

AUCTION GAME

- Players: buyers
- Actions: Each buyer i submits a bid $b_i \in \Theta$
- Outcomes: An outcome is a payment made by each bidder AND an allocation, so

$$O = \{((p_i)_{i=1}^n, x) | \forall i \; p_i \in \Re, x \in I \cup \{seller\}\}$$

- Mapping: mapping from bids to outcomes is a function $g: \Theta^{\mathrm{n}} \rightarrow O$
- Payoffs: For each buyer i, his payoffs $v_i: \Theta \times O \rightarrow \Re$ are of the form

$$(1)\ \forall \theta_i, o = \left(\left(p_j\right)_{j=1}^n, x\right), v\left(\theta_i, \left(\left(p_j\right)_{j=1}^n, x\right)\right) = \theta_i 1_{x=i} - p_i$$

STRATEGIES 1

- Assume buyers behave symmetrically, so a strategy only depends on an individual's information
- Only consider auctions in which all bidders would be willing to participate, so *if* a bidder does not win an auction, he pays nothing
- Focus on strictly increasing bidding strategies $b(.)$ (to be defined later): i.e. for each $\theta, \theta' \in \Theta, \text{s.t.}\, \theta < \theta', b(\theta) < b(\theta')$
- The bidding strategy will be assumed differentiable
- Do these restrictions matter?
- Yes, buyers can bid non-symmetrically, bidding strategies need not be strictly increasing or differentiable, and I rule out R&D as part of the discussion

STRATEGIES AND EQUILIBRIUM

- Definition: A buyer strategy is a function $b\colon \Theta \to \Theta$ where for each $\theta, b(\theta) \in \Theta$ is the bid placed by a buyer with valuation θ
- Definition: An equilibrium is a strategy $b(.)$ such that

i. For every $i, \theta_i, b(\theta_i)$ solves

$$(2)\ U(\theta_i, b) = \max_{b\in\Theta} E\left[v\left(\theta_i, g\left(\left(b(\theta_j)\right)_{j=1}^{n}\right)\right)\right] = \max_{b\in\Theta} \theta_i\, Pr(\, i\ wins|\theta, b) - E(p_i|\theta, b)$$

FIRST PRICE AUCTIONS

- Let us now look at particular of auctions rules
- Definition: A sealed-bid, first price auction with reserve price p, goes as follows

i. Bids must be numbers above p

ii. Buyer submitting highest bid wins the item and pays his bid

iii. If more than 1 buyer submit identical bids, then they win the good with equal odds

iv. Losing buyers pay nothing

EXAMPLE: PART 1

- Suppose that $\forall\theta \in [0,1], F(\theta) = \theta$, p=0, then given that a buyer i has a valuation θ_i, odds he wins the good when he bids b, equals to

  $$(3)\ \Pr(i\ wins|\theta_i) = \Pr\left(\forall j \neq i, \theta_j \leq b^{-1}(b)\right) = \Pr\big(\theta \leq b^{-1}(b)\big)^{n-1} = F[b^{-1}(b)]^{n-1} = b^{-1}(b)^{n-1}$$
  $$= b^{-1}(b)$$

- If he bids b, then his expected payment equals to

  $$(4)\ b\Pr(i\ wins|\theta_i) = bb^{-1}(b)$$

- His expected payoffs when he bids b are then

  $$(5)\ u(b,\theta) = (\theta - b)b^{-1}(b)$$

EXAMPLE: PART 2

- Buyer's problem then solves

$$(6)\ U\big(\theta, b(.)\big) = \max_{b\in[0,1]} (\theta - b)b^{-1}(b)$$

- Q: How do we solve this problem?
- A: Lagrange Equation:

$$(7)\ \mathcal{L}(b, \phi_0, \phi_1) = (\theta - b)b^{-1}(b) + \phi_0 b + \phi_1(1 - b)$$

- FOCs are

$$(8)\ \frac{(\theta - b)}{b'(b)} - b^{-1}(b) + \phi_0 - \phi_1 = 0$$

EXAMPLE: PART 3

- In equilibrium, $b = b(\theta)$, so once we plug in this fact into the FOC, it holds that for $\theta > 0$

$$(9)\ \frac{\left(\theta - b(\theta)\right)}{b'\left(b(\theta)\right)} - b^{-1}\left(b(\theta)\right) = 0$$

- This equation simplifies to

$$(10)\ [\theta - b(\theta)] = \theta b'(\theta)$$

- Further into

$$(11)\ \theta = b(\theta) + \theta b'(\theta) = \frac{d}{d\theta}[\theta b(\theta)]$$

EXAMPLE: PART 4

- We can now integrate the last equation and get

$$(12)\ \frac{\theta^2}{2} = \theta b(\theta)$$

- Thus, the equilibrium bidding strategy is $b(\theta) = \frac{\theta}{2}$.
- I will go back to this expression

SECOND PRICE AUCTIONS

- Definition: A sealed-bid, second price auction with reserve price p, goes as follows

i. Bids must be above p

ii. Buyers submitting highest bid wins the item

iii. If multiple buyers submit identical bids, all buyers win the good with identical odds

iv. If no one else submits a bid, he pays p

v. If some other buyer submits a bid, he pays the second highest bid

vi. Losing buyers pay nothing

KEY FACTS ABOUT SECOND PRICE AUCTIONS

- The strategy $\forall\theta, b(\theta) = \theta$ is an equilibrium. Why?

i. If a buyer has a valuation θ and bids $b > \theta$, then he increases his chances of winning the good, BUT he exposes himself to paying $x > \theta$ and thus he would have been better off not participating

ii. If the buyer with valuation θ bids $b < \theta$, then he expects to be less likely to win AND his actions does not change how much he would pay IF he ends up winning.

- Is this the ONLY possible equilibrium? Not by a long shot.
- Counterexample: Bob and Alice bids against each other, with the same primitives as in the past example, then the following is an equilibrium:

i. Bob always bids 1

ii. Alice Always bids 0.

EXAMPLE: PART 5

- Suppose bidders, instead, participated in a sealed-bid, second price auction with a reserve price of 0
- Valuations are still drawn given CDF $\forall\theta \in [0,1], F(\theta) = \theta$ and there are two buyers
- Suppose buyers play the equilibrium strategy $\forall\theta, b(\theta) = \theta$, then

i. What would be a buyer's expected odds of winning the good?

ii. How much does he expect to pay?

EXAMPLE: PART 6

- Odds of winning just equal odds that the buyer has a valuation above his peer, so for each buyer i

$$(13)\ \forall\theta_i, \Pr(i\ wins|\theta_i) = \Pr(\theta_{-i} \leq \theta_i) = \theta_i$$

- Expected payment, if he wins, equals to the expected valuation of the opponent, so

$$(14)\ E\{\theta_{-i}|\theta_i\} = \int_0^{\theta_i} \frac{x}{\theta_i} dx = \frac{x^2}{2\theta_i} |_0^{\theta_i} = \frac{\theta_i}{2}.$$

EXAMPLE: PART 7

- Note that the buyer with valuation θ

1. Expects to win IF AND ONLY IF he has the highest valuation
2. If he wins, he pays $\frac{\theta}{2}$ in the first-price auction for sure and $\frac{\theta}{2}$ on average in the second-price auction
3. Hence, the buyer expects the same surplus from participating in either auction
4. AND the Seller expects the SAME level of profits.

REVENUE EQUIVALENCE

- This last observation is not an accident, it holds in a rather general setting
- Revenue Equivalence: Consider 2 difference auction rules. Suppose that in BOTH

i. The buyer with the lowest possible valuation gets a payoff of 0

ii. The buyer with the highest valuation wins

Then, both auctions generate the same expected revenues.

- Punchline: If we abstract away from sequential interactions, what determines the returns to an auction is the allocation decision and NOT the particular payment rule choice.

FINAL POINTS:

- Q: But why do we tend to see first price auctions and not second price auctions in the real world?
- A: Akbarpour and Li (2019) show that second price auctions are not credible, meaning that the seller could always say that the second highest valuation is higher than it really was and individual bidders who do not talk to each other cannot credibly show that the seller was lying.
- In fact, the first price auction is the ONLY simultaneous bidding procedure that is credible as describe above
- Furthermore, if we consider sequential bidding procedures, it turns out that English Auctions are the only credible auction procedure.