LECTURE 2

JD R-M

FOR TODAY

i. Expectations
ii. Probabilistic learning
iii. Perfect Competition

EXPECTATIONS 1

- Suppose that we have a finite set of states of the world $\Omega = \{\omega_i\}_{i=1}^n$ and we define probabilities on each event having a vector $\pi = \{\pi_i = \Pr(\omega_i)\}_{i=1}^n$
- A function $x: \Omega \rightarrow \Re$ is called a real-valued, random variable (rv)
- There are several things that one may want to define for a random variable.

1. What should one expect for its "common" realization? We answer this with an expectation as

$$E[x] = \sum_{i=1}^{n} \pi_i x(\omega_i)$$

2. A second question is what should a transformation $f(.)$ average realization be? Well just define a new rv for every $\omega \in \Omega, y(\omega) = f[x(\omega)]$ and then

$$E[f(x)] = E[y] = \sum_{i=1}^{n} \pi_i y(\omega_i)$$

EXPECTATIONS 2

- It is also possible for us to have $\Omega = \Re$ and under some technical conditions, we can denote a function f: $\Re \rightarrow \Re_+$ that is called the probability density function (pdf)
- The function $f(.)$ further satisfies

$$\int_{-\infty}^{\infty} f(\omega) d\omega = 1$$

- If x is then a r.v., we have that

$$E[x] = \int_{-\infty}^{\infty} x(\omega) f(\omega) d\omega \, , E[f(x)] = \int_{-\infty}^{\infty} f[x(\omega)] f(\omega) d\omega$$

EXPECTATIONS 3

- An analogous object to pdf is the Cumulative Density Function (CDF) $F: \Re \rightarrow [0,1]$

$$\forall y \in \Re, F(y) = \Pr(x \leq y) = \int_{-\infty}^{y} f(\omega) d\omega = E[\chi(x \leq y)],$$

Where $\chi(x \geq y) = 1$ if the statement is true and it is otherwise equal to 0

Properties:

1. CDFs are non-decreasing
2. They can always be defined for real valued functions, EVEN when pdfs may not exist
3. Except for a countable set of points in its domain, CDFs must be continuous functions
4. We will use them to model markets.

PROBABILISTIC LEARNING

- Fix a probability space (Ω, Σ, p)
- Assume that one has a family of events $\bar{A} = \left(A_j\right)_{j=1}^{K} \subset \Sigma$ such that

1. For every distinct pair $j, j',\ A_j \cap A_{j'} = \emptyset$
2. For every j, $P\left(A_j\right) > 0,$
3. $\cup_{j=1}^{J} A_j = \Omega.$

- I will call $\bar{A}$ an "experiment".

LEARNING FROM AN EXPERIMENT

- Suppose that one observes a particular event $A \in \bar{A}$, how does one describe how we perceive how likely are each state of nature? Well,

i. If $\omega \in A^c$, then one is certain that ω cannot have occurred.

ii. For all intents and purpose, we deduce that the set of states of nature is A

iii. We now need a probability space with A taking the place of Ω, but we also want it to be "consistent" with what we used to believe.

NEW CONSTRUCTION

- We first define the set of events given A as $\Sigma_{|A} \equiv \{C|C = A \cap B \; for \; some \; B \in \Sigma\}$
- This means that the states of nature which are inconsistent with what we know (i.e. event A) are disregarded
- We then define a probability function $p(.|A)\colon \Sigma_{|\mathrm{A}} \to [0,1]$ as follows

$$(1) \; \forall C \in \Sigma_{|A}, p(C|A) = \frac{p(C)}{p(A)} = \frac{p(A \cap B)}{p(A)} \; for \; C = B \cap A.$$

IMPORTANT OBSERVATION

- Fix some sets $A, B \in \Sigma, A \in \bar{A}$.
- Suppose that one has a different experiment $\bar{B}$ such that $B \in \bar{B}$
- Then we can define $p(.|B)$ as described before, but now notice that

$$(Bayes\ Rule)\ P(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{p(B|A)p(A)}{p(B)}$$

- This is the starting point for modern science.

IN WHAT SENSE ARE THESE PROBABILITIES CONSISTENT?

- Now, we have a collection of probability function $\left(p(.\,|A)\right)_{A\in\bar{A}}$
- Pick some set $B \in \Sigma$, then notice that

$$(2)\ \sum_{A\in\bar{A}} p(A)p(B\cap A|A) = \sum_{A\in\bar{A}} p(A)\left[\frac{p(A\cap B)}{p(A)}\right] = \sum_{A\in\bar{A}} p(A\cap B) = p\big((\cup_{A\in\bar{A}} A)\cap B\big) = p(\Omega\cap B) = p(B)$$

- This is called Bayes consistency: for every event $B \in \Sigma, p(B) = E[p(B|A)\,|\bar{A}]$

PERFECT COMPETITION MODEL

- This class focuses on markets with pathologies affecting the performance
- But their performance relative to what?
- I now develop a model of perfect competition that serves as a maxim of efficiency against whom other markets are to be compared
- A competitive market is one where all agents (both buyers and sellers) are irrelevant to the outcome as a whole
- But I will write it in a more game theoric fashion to prepare the class for tnings to come.

ENVIRONMENT 1

- Players: Unit mass of atomic market participants (the set of players is $I = [0,1]$)
- Role types: A market participant can be either buyers or sellers
- Probability of role types: The odds that a market participant is a buyer is $\alpha \in (0,1)$

****Role types are drawn pairwise independently.

- Actions:

i. Buyers: Each buyer i can either buy $a_i = 1$ or not $a_i = 0$.

ii. Seller: Each seller j can either sell $b_i = 1$ or keep his good $b_i = 0$

ENVIRONMENT 2

- Payoffs: Each market participant has a valuation for the good $\theta \in \Re_+$ and I assume that it is distributed given a CDF $F(.)$
- Valuations are drawn pairwise independently (Hic Sunt Dacrones)

i. Suppose that the price ends up being $p \in \Re_+$, then payoff are

$$(Buyer\ Payoffs)\ u_i(\theta,p) = \begin{cases} 0\ if\ a_i = 0 \\ \theta - p\ if\ a_i = 1 \end{cases} = a_i(\theta - p)$$

$$(Seller\ payoffs)\ \ \pi_j(\theta,p) = \begin{cases} \theta\ if\ b_i = 0 \\ p\ if\ b_i = 1 \end{cases} = b_i p + (1 - b_i)p$$

TIMING

- The market game occurs in the following order

i. Nature draws a role type and a valuation θ

ii. Nature privately informs each buyer of their own role and type

iii. Market participants make choices simultaneously

iv. Agents make choices and clear the market.

STRATEGIES 1

- Heuristically: A strategy is a function from what a player knows to what he can do.
- I assume that market participants follow an anonymous strategy. This means that any two buyers with the same information take the same action
- Q: But what is the information available to a market participant?
- A: He knows whether he is a buyer or seller as wellas his valuation

STRATEGIES 2

- A pure buyer strategy is a function $\sigma_b: \Re_+ \to \{0,1\}$
- A pure seller strategy is a function $\sigma_s: \Re_+ \to \{0,1\}$

****Strategies ARE ALWAYS FUNCTIONS.

BUYER PROBLEM

- Assume that a buyer observes a type θ and conjectures a price p, then he picks an action a_i solving

$$(3)\ U(\theta, p) = \max_{a \in \{0,1\}} a(\theta - p)$$

SELLER PROBLEM

- Assume that a seller observes a type θ and conjectures a price p, then he picks an action b_i solving

$$(4)\ \pi(\theta, p) = \max_{b \in \{0,1\}} bp + (1-b)\theta$$

EQUILIBRIUM

- An equilibrium is a triple (σ_s, σ_b, p) such that for every θ,

i. $\sigma_S(\theta)$ solves $\pi(\theta, p)$,

ii. $\sigma_b(\theta)$ solves $u(\theta, p)$

iii. Markets clear: $\alpha E_F[\sigma_b(\theta)] = (1-\alpha)E_F[\sigma_s(\theta)]$

CHARACTERIZATION 1: BUYERS

- Observe that given a conjectured price p, the buyer strategy strategy is

$$(5)\ \sigma_p(\theta) = \begin{cases} 1\ if\ \theta \geq p \\ 0\ otherwise \end{cases} = 1 - \chi(\theta \leq p)$$

CHARACTERIZATION 2: SELLERS

- Observe that given a conjectured price p, the buyer strategy strategy is

$$(5)\ \sigma_s(\theta) = \begin{cases} 1\ if\ \theta \leq p \\ 0\ otherwise \end{cases} = \chi(\theta \leq p)$$

CHARACTERIZATION 3: AGGREGATES

- Total demand is then $\alpha E_F[\sigma_b(\theta)] = \alpha E_F[1 - \chi(\theta \leq p)] = \alpha[1 - F(p)]$
- Total Surplus is $(1 - \alpha)E_F[\sigma_s(\theta)]$= $(1 - \alpha)E_F[\chi(\theta \leq p)] = (1 - \alpha)F(p)$
- By the market clearing condition, the price is whichever value solving

$$\alpha[1 - F(p)] = (1 - \alpha)F(p)$$

Or that

$$(6)\ F(p) = \alpha$$

PUNCHLINE

- The price ensure that the price equalizes the share of buyers with the share of sellers trading
- In other words, prices serve to move goods from those who don't value goods to those who do.

NEXT CLASS

- Welfare Analysis.