The Generalized Coase Conjecture

By Jorge D. Ramos-Mercado*

This paper studies the Coase Conjecture in auctions settings with interdependent values (IV). In each period, the seller credibly runs a second-price auction with a reserve price. But if the item fails to sell, the seller cannot commit to keep the good in the future. I find that the equilibrium revenues are unique and independent of how often the seller offers his good to potential buyers. Intuitively, when the item fails to sell, buyers screen their peers and consequently lower their valuation. This ensures that, in all equilibria, the seller can only auction the good a finite number of times. Lastly, I prove that the equilibrium revenues equal immediately running the revenue maximizing auction after which the seller cannot gainfully re-offer his good.

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It is well understood that a seller's sequential rationality limits his attainable profits. Coase (1972) argued that a durable good monopolist expects high valuation buyers are more likely to buy his good than their low valuation peers. In response, a seller, who cannot initially pre-commit to a price schedule, lowers prices so as to attract further purchases. The resulting price schedule, however, persuades some high valuation buyers to delay their purchase decisions. Coase further predicted that as agents interact increasingly, the buyer's profits converge to 0. Liu et al (2019) further proved that this conjecture generically holds when the seller auctions his good.

Ramos-Mercado (2022), nonetheless, prove that the Coase conjecture fails when buyer valuations have an interdependent component. For example, wildcatters

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participating in a drilling rights auction derive estimates for a plot of land's oil reserves and expect that their peers estimates are also informative of the reserves in question. The paper in question proves that learning among buyers limits the number of times that an item can be gainfully offered by the seller and, hence, the difference between the profits attained under full commitment and the unique equilibrium profits is bounded. Indeed, it is possible for the seller's equilibrium profits to precisely equal to those attained with full commitment.

In this paper, I study second-price auctions with interdependent value (IV) settings. I establish a generalized Coase Conjecture (GCC, from henceforth). Ramos-Mercado (2022) focuses on the way learning among buyers allows the seller to extract a larger share of full commitment profits than the seller could extract in a comparable setting with private values. However, buyers are assumed to have multidimensional types and this precludes an intuitive extension of the Coase conjecture. To this end, this paper focuses on the case where buyer valuations are one-dimensional. I prove that the seller's equilibrium profits are unique and equal immediately running the profit maximizing auction after which the seller cannot gainfully re-auction his good.

In Ramos-Mercado (2022), I prove that this is a lower bound on the seller's equilibrium profits but it need not bind. This is because that paper's main contribution is that learning among buyer yields the principle of progressive pessimism; i.e. over time, prior distributions of valuations likelihood ratio dominate their posterior distributions. Consequently, the seller extracts an increasing share of surplus from buyers and, in second-price auctions, this implies that buyers are willing to bid an increasing share of their valuations. This is because learning among buyers from a lack of trade ameliorates the winner's curse since buyers expect that winning the good is less informative of their peer's interdependent value components. When buyers only observe a one-dimensional IV type, however, their bidding strategy is unencumbered by learning.

When types are one-dimensional and interdependent, the seller cannot gain from screening his buyers in the same manner as when their types are multi-dimensional. Indeed, I prove that no screening among buyers allows the seller to attain more profits than immediately running the revenue maximizing auction after which the seller can credibly commit to keeping his good. This allows me to state a generalized intuition for the Coase conjecture. When valuations are interdependent, buyers screen each other and consequently destroy the surplus that could have been traded among buyers. As a consequence, the seller is prompt to immediately transact with every buyer who values the good more than himself and would not do so upon learning that the item failed to sell—regardless of how often he trades his good. These are the analogous profits to immediately running an efficient auction. Consequently, the difference between the Coase conjecture with interdependent rather than private values is that interdependence ensures that the seller's revenues are driven to their revenue floor without mention to how often agent interact.

The rest of the paper is organized as follows. I first present the model. I then establish the generalized Coase Conjecture. Lastly, I discuss the results and conclude.

I. Model

A seller offers a single, indivisible good to $n \geq 2$ buyers. Each buyer i privately observes a type $x_i \sim U[0,1]$. Types $(x_i) \in \mathcal{X} \equiv [0,1]^n$ are drawn iid and each buyer i's payoff from owning the good is $u(x_i, x_{-i})$ where $u : \mathcal{X} \to [0,1]$ satisfies assumptions 1.

ASSUMPTIONS 1: *The payoff function $u(\cdot)$ is strictly increasing, continuously differentiable, $u(0,0,\ldots,0) = 0, u(1,1,\ldots,1) = 1$, and it is symmetric; i.e. for each x_i, x_{-i}, and permutations $\sigma(.)$ on x_{-i}, it holds that $u(x_i, x_{-i}) = u(x_i, \sigma(x_{-i}))$.*

The seller, for his part, has a valuation of the good of $\theta_s \in (0,1)$. As in Ramos-Mercado (2022), this assumption will ensure that the seller only re-offers his good a finite number of times when buyers have interdependent values but this would not occur when buyers have private values.

I now describe the timing of the game. Nature first draws the types (x_i) and privately informs each buyer i of their own x_i. Next, in each period $t = 0, 1, \ldots$, the seller first posts a reserve price p_t. Buyers then decide to wait or submit a bid above p_t. If no buyer bids, the game continues to period $t+1$. Otherwise, the game ends, the buyer submitting the highest bid wins the auction, and either pays the second highest bid or p_t provided that no other buyer placed a bid. Moreover, if multiple buyers submit the highest bid, then each wins with equal probabilities. Lastly, if buyer i wins the item in period t and must pay $p_t^i \geq p_t$, payoffs are

i. Buyer i: $\delta^t(u(x_i, x_{-i}) - p_t^i)$ for a common discount factor $\delta \in (0, 1)$,

ii. Buyers $j \neq i$: 0

iii. Seller: $(1 - \delta^t)\theta_s + \delta^t p_t^i$.

I lastly define strategies and equilibrium. First define for each period t, H_t to be the set of histories $h_t = \{p_s\}_{s=0}^{t-1}$ and let h_0 be some null history. A seller strategy is then a collection of functions $(p_t) \quad \forall t, p_t : H_t \to [0, 1]$ denoting reserve prices[1]. Next, I assume that buyers play a symmetric strategy and when indifferent between bidding and waiting, they bid. Their strategy is a collection $(b_t), \quad \forall t, b_t : H_{t+1} \times [0, 1] \to \{wait\} \cup [0, 1]$, such that for each period t, history $h_{t+1} = (h_t, p_t)$, and type x_i , $b(h_{t+1}, x_i)$ denotes a decision to wait, i.e. $b(h_{t+1}, x_i) = wait$, or an admissible bid: $b(h_{t+1}, x_i) \in [p_t, 1]$. Beliefs, for their part, are history dependent joint measures on and values (x_i). Now, a Bayes Perfect Equilibrium (PBE) consists of seller and buyer strategies as well as beliefs s.t. given beliefs and the strategies, the seller and buyers behave sequentially rational

[1] All functions are be assumed Lebesgue measurable. Furthermore, Liu et al (2019), Fundenberg, Levine, and Tirole (1985), and others find that almost surely neither the seller or buyers play a mixed strategy.

in every subgame following each history and beliefs are derived from strategies on the equilibrium path.

II. Generalized Coase Theorem.

In this section, I state my main result. First, I prove an auxiliary result. It states that buyers follow a threshold strategy. Heuristically,this means that in each period, there exists a cutoff value and every buyer who observed a type above this level participates in the current auction so long as the good remains unsold. Next, I prove that the seller's equilibrium revenue is bounded below by running a subset of auctions after which the seller can commit to not re-auction the good. Lastly, I prove that the equilibrium is essentially unique and that revenues precisely equal the aforementioned lower bound.

The first step is to characterize buyers' equilibrium behavior. First, I claim that buyers follow a threshold participation rule.

DEFINITION 2 (Threshold Strategy): *A buyer i is said to play a threshold strategy if there exist a collection of functions functions $(u_t), \forall t\ u_t : H_{t+1} \to [0, 1]$, such that for every period t and history h_{t+1}, each buyer i participates iff $x_i \geq u_t(h_{t+1})$.*

I will suppress history notation from henceforth in order to ease exposition. Next, the fact that buyers must follow a threshold strategy (u_t) implies that if the item failed to sell by period t, then each buyer i learns by period $t + 1$ that other buyers $j \neq i$ have types $x_j \leq u_t$. Consequently, each buyer i expects that his peers $j \neq i$ has a type $x_j \sim U[0, u_t]$. I can now characterize buyers' equilibrium strategies in the following lemma.

LEMMA 3: *In every PBE, buyers play a threshold strategy (u_t) and when a buyer i participates at an auction, he bids*

$$w_t^i \equiv v(x_i, x_i) = E[u(x_i, x_{-i}) | \forall j, x_j \leq x_i]. \tag{1}$$

Lastly, each buyer i has a valuation in period t of

$$v_t^i \equiv v(x_i, u_t) = E[u(x_i, x_{-i})|\forall j, x_j \leq u_t]. \tag{2}$$

I delegate proofs to the appendix, but describe its implication here since the proof is standard. This lemma states two assertions of interest. First, buyers decisions to participate in auctions satisfy a natural cutoff rule in every equilibrium, which is precisely what occurs when buyers have private values—see Ausubel and Deneckere (1989), Fundenberg et al (1985), and Liu et al (2019). It states that buyers who observes higher types are more likely to participate than their peers observing low types. This is an important feature as it preserves the intuition at the core of the Coase conjecture.

Next, each buyer i does not bid their valuation v_t^i and instead, they bid the valuation conditional on winning i.e. w_t^i. This is because each buyer i expects his peers bidding strategy to be strictly increasing in their type. Hence, if buyer i wins an auction, he learns that he observed the highest type. It is important to notice that this contrast to buyer's bidding behavior in Ramos-Mercado (2022); because when types are multidimensional, each buyer i's beliefs regarding their peer j's interdependent component is further compressed to the lower portion of the support.

I now move on the describe the paper's main lemma, i.e. the Generalized Coase Conjecture. First, I claim that the seller's equilibrium revenues are unique, regardless of δ. In the game theory literature, this feature of the game is known as the equilibrium being essentially unique. But first, define $\phi(\cdot)$ to denote the ironed out virtual value of initial valuations, i.e. the virtual value is for each type x_i is $v(x_i, x_i) - (1 - x_i)v_1(x_i, x_i)$. Further define $F^2(\cdot)$ as CDF for the second highest value out of n random variables distributed uniformly between 0 and 1. Lastly, let r_0 denote the seller's period 0 expected revenues. I can now derive the theorem.

THEOREM 4: *The equilibrium is essentially unique and the seller's revenues equal to*

$$r_0 = \max_{\theta^* \in [0,1]} \int_{\theta^*}^{1} \phi(y) dF^2(y) \;\; s.t. \;\; v(\theta^*, \theta^*) \le \theta_s. \tag{3}$$

All proofs are delegated to the appendix, but I sketch the argument here. There are four steps. First, I establish the existence of a Markov Equilibrium closely following the argument in Ausubel and Denekere (1989) as well as in Liu et al (2018). I prove the existence of a Markov equilibrium where the buyers' state variable is the current period and the seller strategy's state variable is a tight upper bound on the values held by buyers. Notice that if buyers follow a threshold strategy (u_t) and the game continues to period $t+1$, then it must be the case that for each buyer i, it holds that all other agent's beliefs regarding i's values only deduce that $x_i \sim U[0, u_t]$. Next, I further show that if a Markov equilibrium can be define on some set $[0, X]$ for $X \in (0, 1]$, then it can be extended to $[0, 1]$ in finitely many steps.

Next, I show that there exists a deterministic period $T < \infty$ after which no buyer places a bid almost surely. I prove this assertion by showing that if by some period t it holds that the maximum type observed by buyers is $u_t, v(u_t, u_t) > \theta_s$, then by period $t + \tau$ it must be that $u_t - u_{t+\tau} > \epsilon$ for some fixed $\tau \in \{1, 2, \cdots\}$ and small $\epsilon > 0$. Otherwise, the seller could immediately post a reserve price p^* which would elicit buyers with types x_i greater than a cutoff $\theta^*(\le u_t)$ to bid. And for each buyer j whose type $x_j \le \theta^*$, it holds that $v(x_j, \theta^*) \le \theta_s$. Posting such price ensures that the seller does not re-auction his good and is a revenue floor which would net the seller higher revenues than what he could attain in equilibrium. This is a contradiction and hence by some period T, it must be that buyers observed types below some value u_T such that $v(x_i, u_T) \le \theta_s$ for each $x_i \le u_T$. Lastly, since actions are commonly observed, then all equilibria can be derived via backwards induction and the payoffs are essentially unique.

III. Discussion and Conclusion

I now discuss the implications of the result and conclude the paper. The lemma states that, regardless of time preferences, the equilibrium revenues are unique and equal to running the revenue maximizing static auction after which the seller *can* guarantee that he will not re-offer his good. This implies that in the presence of interdependence, the generalization states that screening among buyers forces the seller to avoid re-offering his good. Consequently, it is no longer the case that over time, the seller exerts a negative externality to his past self. Instead, learning among buyers destroys value and prevents the seller from gainfully screening the buyers himself regardless of how often the agents interact.

I conclude the paper by making several technical clarifications of note. First, the assumption that the seller runs second-price auctions is only made for exposition and the argument extends to the case that he runs English auctions and first-price auctions. The only gain from presenting results using second-price auctions is that the bidding strategy is simpler to describe. Nonetheless, all substantive results persist in such auction environments. Next, it should be noted that, in the appendix of Ramos-Mercado (2022), I already establish that in the case of durable good markets with interdependent values, the seller can extract all rents from buyers. Thus, even the restriction to an auction setting does not determine the validity of the Generalized Coase Conjecture.

The following technical results depend on some of the technical assumptions made. First, I assume that types are drawn uniformly for simplicity. If one allows the payoff function $u(\cdot)$ to be sufficiently general, then it can embed the cases where types are drawn by a more general CDF $F(\cdot)$ that admits a PDF. The assumptions that have more bite is symmetry. This is because if each buyer had a payoff function $u_i(\cdot)$ or his type x_i were distributed asymmetrically, then the model loses its tractable description. That being said, the main intuition remains in spite of these assertions.

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PROOFS.

Proof of Lemma 3.

Proof

Fix some PBE,then the fact that buyers play a threshold strategy (u_t) follows from chapter 10.2.4's Lemma 10.1 in Fundenberg and Tirole (1991); meanwhile, the fact that buyers truthfully report their valuations in a standard auction follows from Myerson (1981). We are now left to establish beliefs. Fix t, h_t, i, x_i, then buyer i's valuation at the history in question equals $f(x_i)E[u(x_{-i})|h_t, x_i]$. But after such history h_t, buyer i solely concludes that for every other buyer $j \neq i$, it must be the case that $x_j \leq u_t(h_t)$. Thus, their valuation equals $f(x_i)E[g(x_{-i})|\forall j \neq i, x_j \leq u_t(h_t)]$. This concludes the proof.

□

B1. Proof of Theorem 4

Proof

This proof has three parts. First, I construct a Markov equilibrium where the buyers' threshold strategy depends on the current price and the seller's strategy depends on the maximum value that a buyer could have received x. Secondly, I prove that the game essentially terminates in finite time and that all PBE are essentially unique. Lastly, I find a closed form expression for the auction at hand.

MARKOV EQUILIBRIUM CONSTRUCTION.. — I claim that there exists a Markov equilibrium as described above characterized by by a price function p and a buyer threshold function w. If such equilibrium exists, then when the maximal value received is y, for each x_i it holds that $E[u(x_i, x_{-i})|x_i, \forall j \neq i, x_j \leq y] = v(x, y)$.

Next, if a buyer, with value x, is indifferent between bidding immediately and waiting, he expects to win only if he is almost surely (as) the *only* bidder and pays the current reserve price of p; meanwhile if he waits, he expects to win as in the subsequent period. With a probability of $\{x'(x)/x\}^{n-1}$, he is the only bidder and receives the subsequent reserve price of p' and otherwise pays the highest valuation among the other $n-1$ bidders. Thus, his indifference implies that

$$\text{(B1)} \quad v(x,y) - p = \delta\left\{v(x,x) - p'\left[\frac{w(x)}{x}\right]^{n-1} - \int_{w(x)}^{x} \frac{(n-1)v(z,x)}{z}\left[\frac{z}{x}\right]^{n-1} dz\right\}.$$

Next, given some maximal value $y \in [0,1]$, the seller might as well assume that prices are given by B1 and might as well pick a maximal values of buyers who wait, namely $x \in [0,y]$. If he sells the good, then his revenues follow directly from the Vickrey allocation rules and otherwise he has the option to re-offer the good in the subsequent period and net the optimal revenue given the upper bound x: i.e. $r(x)$. Thus, his payoffs are given by the value function

$$\text{(B2)} \quad r(y) = \max_{x\in[0,y]} np(x)(y-x)x^{n-1} + \int_{x}^{y} \frac{v(z,y)(n-1)nz^{n-2}(1-z)}{2} dz + \delta r(x).$$

Next, I prove that $r(.)$ is a strictly increasing and Lipschitz continuous. Observe that both results are immediate generalizations of Ausubel and Denekere (1989) and Liu et al (2019).

LEMMA 5: *Suppose that $\{p,w\}$ form a Markov equilibrium with associated functions r, then*

$$\text{(B3)} \qquad \forall 0 \le y' < y \le 1, \;\; 0 < r(y') - r(y) \le n(y - y').$$

Proof

Suppose that $\{p,w\}$ form a Markov equilibrium with associated functions r and pick some pair $y, y' \in [0,1]$ satisfying $0 \le y' < y \le 1$. Then, first note that

(B4)

$$\begin{aligned} r(y) &= \max_{x'\in[0,y]} p(x')n(y-x')x'^{n-1} \\ &+ \int_{x'}^{y} zv(y)\left[\frac{(n-1)nz^{n-2}(1-z)}{2}\right]dz + \delta r(x') \\ &\geq \max_{x'\in[0,y']} p(x')n(y-x')x'^{n-1} \\ &+ \int_{x'}^{y} zv(y)\left[\frac{(n-1)nz^{n-2}(1-z)}{2}\right]dz + \delta r(x') \\ &> \max_{x'\in[0,y']} p(x')n(y-x')x'^{n-1} + \int_{x'}^{y'} zv(y')\left[\frac{(n-1)nz^{n-2}(1-z)}{2}\right]dz + \delta r(x') \\ &= r(y'). \end{aligned}$$

Next, it is clear that $r(y) \leq (y^n - y'^n) + r(y')$ as the additional difference would be equivalent to being above to fully extract maximal possible rents from the winning buyer. Thus, it holds

(B5) $$r(y) - r(y') \leq y^n - y'^n \leq n(y - y').$$

This concludes the proof.

□

Now that we have established the Lipschitz condition mentioned above. I establish that is for some $X \in (0,1)$ it is possible to derive an equilibrium $\{p_X, w_X\}$ on the support $[0, X]$, then one can extend such equilibria to $[0,1]$.

LEMMA 6: *Suppose that for some $X \in (0,1]$ a pair of functions $\{p_X, w_X\}$ supports an equilibrium on $[0,X]$, then there exists some equilibrium on $[0,1]$ characterized by $\{p, w\}$ such that supp $p =$ supp p_X.*

Proof

I present an iterative argument extending the equilibrium over $[0,1]$. Suppose that for some $X \in (0,1]$ there exists a Markov Equilibrium on $[0,X]$ given by a pair of functions (p_X, w_X) and associated revenue function r_X. Define $Y_0 = X$

and for each $m \geq 1$ let $Y_m = \min\{1, \sqrt[n]{Y_{m-1} + (1 - \delta r_X(X)}\}$. First, it holds that for $x \in [0, X]$, the seller equilibrium might as well equal to (p_X, w_X). For $y \in [X, Y_m]$, revenues equal to

(B6)
$$r_{Y_m}(y) = \max_{x \in [0,X]} p_X(x) n(y-x)x^{n-1} + \int_x^y \frac{v(z,y)(n-1)nz^{n-2}(1-z)}{2} dz + \delta r_Y(x).$$

and the thresholds and prices satisfy

(B7)
$$v(x,y) - p_{Y_1}(x) = \delta \left[v(x,x) - P_X[\inf \operatorname{argmax} r_{Y_m}(x)] \left[\frac{\inf \operatorname{argmax} r_{Y_m}(x)}{x} \right]^{n-1} - \int_{\inf \operatorname{argmax} r_{Y_m}(x)}^{x} \frac{(n-1)z^{n-2} v(z,x)}{x^{n-1}} dz \right].$$

Equation B6 disregards the possibility that the seller picks a value $x \in [X, Y_m]$. I claim that this is without loss of generality since for each $x \in [X, y], y \in [X, Y_m]$, it holds that

(B8)
$$\begin{aligned} p_X(x) n(y-x)x^{n-1} + \int_x^y \frac{zv(y)(n-1)nz^{n-2}(1-z)}{2} dz + \delta r_{Y_m}(x) \\ \leq y^n - x^n + \delta r_{Y_m}(x) \\ \leq (1-\delta) R_X(X) + \delta r_{Y_1}(x) \\ = (1-\delta) R_{Y_m}(X) + \delta r_{Y_1}(x) \\ \leq r_{Y_m}(x). \end{aligned}$$

Notice that the first inequality uses the Lipschitz condition on the revenue gains; in the second the definition that $Y = \min\{1, \sqrt[n]{X + (1 - \delta r_X(X)}\}$, and in the third that revenues are non-decreasing in the upper bound. This implies that one can extend the equilibrium to $[0, Y_m]$ and for each maximal value $y \in [0, Y_m]$, seller picks a maximal value among delaying buyers of $x \in [0, X]$. This iteration terminates in finitely many steps and concludes this proof. □

Next, I find an $\underline{\mathrm{x}} \in (0,1)$ and find a Markov Equilibrium on $[0, \underline{\mathrm{x}}]$.

LEMMA 7: *For some $\underline{x} \in (0,1)$ there exists Markov Equilibrium characterized by a pair of function $\{p_{\underline{x}}, w_{\underline{x}}\}$ such that for each $y \in [0, \underline{x}], p(y) = v(y, y)$.*

Proof

Since u is a strictly increasing, continuous, and for each $x \in [0,1], 0 \in Int[supp\ u(x, .)]$, then $v(.,.)$ satisfies both conditions. If $v(1,1) \leq 0$, then the lemma's statement is moot as $v(1,1)$ is an upper bound on the seller returns and lies below the seller's return from keeping the good: i.e. 0. Otherwise, $v(1,1) > 0$ and observe that for each $y \in [0,1]^{n-1}$, it holds that $u(0, y) < 0$, so $v(0, y) < 0$ for each $y \in [0,1]$. This implies that $v(0,0) < 0$. Since v is strictly increasing and continuous, then there exists a unique $\underline{\mathrm{x}} \in (0,1)$ such that $v(\underline{\mathrm{x}}, \underline{\mathrm{x}}) = 0$.

Next, I claim that the pair of functions in the prompt form an equilibrium on $[0, \underline{\mathrm{x}}]$. Observe that as v is a strictly increasing function and for each $y \in [0, \bar{x}]$, seller revenues are bounded above by $v(y, y) \leq 0$, then the seller prefers keeping the item rather than offering the good and the strategy profile in the prompt ensure this occurs. This concludes the proof.

□

Game effecitvely ends in finite time.. — I now establish that in all PBE, the market effectively terminates in finite time. Since buyers play a threshold strategy, I claim that for a sufficiently large, but finite T, it holds that $u_t \leq \underline{\mathrm{x}}$. Quickly notice that if $\underline{\mathrm{x}} \geq 1$, then this result is moot at the seller would never offer the good as he almost surely has the highest valuation of the good from the beginning.

LEMMA 8: *For every PBE characterized by a pair (u_t, p_t), there exists a period $T < \infty$ such that for every period $s \geq T$, $w_s = u_T$.*

Proof Suppose for contradiction that there exists a PBE such that on the equilibrium path and at some period t, it holds that the maximal value held by consumers is $u_t > \underline{x}$ where $v(\underline{x}, \underline{x}) = \theta_s$ and for some small $\epsilon > 0$, it holds that $u_t - u_{t+s} < \epsilon < u_t - \underline{x}$ for each $s \in \{1, 2, \cdots \tau\}$ and τ large. define $\pi^e(x) \gg 0$ as the expected return from running a static, efficient auction and $\pi(x)$ the optimal static auction. Then, the equilibrium payoffs in period t, namely r_t^σ, satisfy

$$r_t^\sigma \leq v(u_t, u_t)[\epsilon + \delta^\tau] \leq v(1, 1)(\epsilon + \delta^\tau) \tag{B9}$$

For sufficiently small ϵ and sufficiently large $\tau > 0$, it holds that in period t

$$\begin{aligned} r_t^\sigma &\leq v(1, 1)[\epsilon + \delta^\tau] \\ &< v(\underline{x}, u_t)(u_t - \underline{x})\underline{x}^{n-1} + \int_{\underline{x}}^{u_t} \frac{(n-1)v(z, u_t)}{z} \left[\frac{z}{x}\right]^{n-1} dz \end{aligned} \tag{B10}$$

where the upper bound at hand equals the payoff the seller receives from picking a price $p_t = v(\underline{x}, u_t)$ and hence $w_{t+1} = \underline{x}$. Observe that the seller can precisely attain such payoff because in the following subgame, beliefs state that buyer values lies below $\underline{x}$ and hence the seller never re-offers the good. Next, observe that by construction of the upper bound at hand, we can define for each $\epsilon > 0$ and period $\tau(\epsilon)$ such that for any period t and $u_t \in [\underline{x}, 1]$, it must be the case that $w_{t+\tau(\epsilon)} < u_t - \epsilon$ almost surely. Thus, for any small $\epsilon > 0$ and corresponding wait $\tau(\epsilon)$, it holds that the game effective ends before a period $T(\epsilon) = \tau(\epsilon)\lceil(1 - \underline{x})/\epsilon\rceil$. This concludes the proof.

□

$$r_t^\sigma \leq v(u_t, u_t)\epsilon + \delta^\tau < \pi^e(u_t)$$

Now, since actions are perfectly observable and the game effectively ends in finite time, then all PBE can be characterized via backwards induction and must be payoff equivalent to the Markov equilibrium previously characterized.

Equilibrium Revenues. — Next, I characterize the revenues. Since all equilibria yield the seller with the same level of revenue, it suffices for us to estimate the revenues in the particular equilibrium characterized. Notice that in period 0, the seller picks a price $p = v(y, 1)$ for some $y \leq \underline{x}$ and thus if a there is no trade by period 0, the market effectively ends afterwards. This implies that the seller's revenues are equal to running a static, Vickrey auction with a reserve price of $p_0 = v(y, 1), y \leq \underline{x}$. In such auction, a buyer with valuation $x \in [0, 1]$, can choose which valuation to bid and when he expects his peers to truthfully submit their bids, his bidding problem is

(B11)

$$CS(x) = \max_{w \in [0,1]} 1_{w \geq y} \left[w^{n-1} v(x, 1) - v(y, 1) y^{n-1} - \int_y^w \frac{n(n+1) z^{n-2} (1-z) v(z, 1)}{2} dz \right].$$

In equilibrium, the buyer, himself prefers to truthfully bid and since u is Lipschitz, then v is Lipschitz and thus it is absolutely continuous in its arguments and the partial derivatives almost surely exists and are uniformly bounded. This implies that Corollary 1 in Milgrom and Segal (2002) holds and CS satisfies the equation

(B12)
$$\forall x \in [0, 1], CS(x) = CS(0) + \int_0^x 1_{z \leq y} z^{n-1} v_1(z, 1) dz$$

Notice that at $x = 0$, the buyer truthfully bids that he has $x \leq y$ and $CS(0) = 0$, so the equation above becomes

(B13)
$$\forall x \in [0, 1], CS(x) = \begin{cases} \int_y^x z^{n-1} v_1(z, 1) dz & if \;\; x \geq y \\ 0 & if \;\; x < y \end{cases}$$

Next, define $\pi(x)$ to be the expected payment a buyer with a valuation of x makes, then consumer surplus also equals to $x^{n-1} v(x, 1) - \pi(x)$ and thus the expected payment equals to

(B14)
$$\pi(x) = \begin{cases} x^{n-1} v(x, 1) - \int_y^x z^{n-1} v_1(z, 1) dz & if \;\; x \geq y \\ 0 & if \;\; x < y \end{cases}$$

The seller's revenue is then the expected payment from the n buyers or

$$
\begin{aligned}
r_0 &= \int_0^1 nx^{n-1} 1_{x\geq y}[v(x,1) - CS(x)]dx \\
&= \int_y^1 nx^{n-1} v(x,1) - n\int_y^x z^{n-1} v_1(z,1)dzdx \\
\int_y^1 nx^{n-1}v(x,1)dx &- \int_y^1 x^{n-1}\int_y^x z^{n-1}v_1(z,1)dzdx \\
&= \int_y^1 nx^{n-1}xv(1)dx - n\int_y^1\int_x^1 dznx^{n-1}v_1(x,1)dx \\
&= \int_y^1 nx^{n-1}v(x,1)dx - \int_y^1 (1-x)nx^{n-1}v_1(x,1)dx \\
&= \int_y^1 nx^{n-1}[v(x,1) - v(1)(1-x)]dx \\
&= \int_y^1 nx^{n-1}\phi(x,1)dx
\end{aligned} \tag{B15}
$$

I claim that $\phi(.,y)$ for each $y \in [\underline{\mathrm{x}}, 1]$ is strictly increasing.

LEMMA 9: *For every $y \in [\underline{x}, 1]$, $\phi(.,y)$ is strictly increasing on the support $[0,y]$.*

Proof

Fix some $y \in [\underline{\mathrm{x}}, 1]$ and pick a pair $x, x' \in [0,y]$ such that $x' < x$. Then, as for each $w \in [0,1]^{n-1}$ $u_1(.,w)$ is non-increasing, then $v_1(.,y)$ is non-increasing and $v_1(x',y) \geq v_1(x,y)$. Next, $v_1(x',y)(1-x') \geq v_1(x,y)(1-x') > v_1(x,y)(1-x)$ and then $-v_1(x',y)(1-x') < -v_1(x,y)(1-x)$. Hence, it holds that $v(x',y) - v_1(x',y)(1-x') < v(x,y) - v_1(x,y)(1-x)$ or equivalently that $\phi(x',y) < \phi(x,y)$. This concludes the proof.

□

Further notice that for each $y \in [0,1]$, there exists a unique $x_y \in (0,y)$ such that $v(x_y,y) = 0$ and $\phi(x_y,y) < v(x_y,y) = 0$. This implies that characterizing the optimal y is clear. Since $\phi(1,1) = v(1,1) > 0$ and for some x_* it holds that $\phi(x_*,1) < 0$, then given that $\phi(.,1)$ is the linear combination of continuous functions defined on a compact set of $[0,1]$: the intermediate value theorem holds and implies there exists an $x^* \in (x_*, 1)$ such that $\phi(x^*, 1) = 0$. Since $\phi(., 1)$ is a strictly increasing function, then x^* is uniquely defined. Now, it is clear what the revenue maximizing auction would be chosen. If $x^* \leq \underline{x}$, then the seller posts a reserve price of $p_0 = v(x^*, 1)$; otherwise, $p_0 = v(\underline{x}, 1)$ as revenues would be strictly increasing with respect to y on $[0, \underline{x}]$. Thus, the seller's revenues equal to

$$r_0 = \int_{\underline{x} \downarrow x^*}^{1} n x^{n-1} \phi(x, 1) dx \tag{B16}$$

This concludes the proof.

□