LECTURE 1

JD R-M

WELCOME TO IO

- For today:

i. Class overview

ii. Optimization

iii. Probability Theory

CLASS TOPICS:

- Game theory market foundations
- Perfect Competition
- Monopolies
- Auctions
- Oligopolistic Competition
- Cartel Behavior
- Communication in markets

ORGANIZATION

1. Math and Probability Review
2. Perfect Competition (ideal markets)
3. Monopolies (Standard Monopoly, Price Discrimination, But why are they even legal?!)
4. Auctions (types of auctions, real-world auctions, optimal auctions, interdependence)
5. Monopolies are Auctions with potentially many buyers
6. Oligopolies (Cournot, Bertrand, Stackelberg, Entry Deterrence)
7. Cartels (Repeated interactions, Information, and Communication)

GRADING

i. 4 Problem Sets (10 points each)

ii. 1 Midterm (20 points)

iii. 1 Project (20 points)

iv. 1 in class Final (20 points)

PROBLEM SETS

- Must work in teams of 3-5
- Work must be typed up (including derivations)
- Late works receive an automatic 0
- Show your work
- They are long and tricky, so do them with anticipation AND come talk to me with questions (I seldom bite people)
- Exam questions are re-writes of problem set questions.

MIDTERM AND FINAL

- 2 (long) questions to be done in 75 minutes
- Closed notes
- Comprehensive of all class material
- Word of advice: make-up exams always tend to be slightly harder than the original, so please avoid them as much as possible.

PROJECT

- Warren Buffet hires you and asks you to find a company to buy
- Pick a firm, analyze its industry, finances, and prospects
- Present your pick to class and write a report rationalize your choice
- BUT there’s a catch. Warrant Buffet does not know or care about the formalism I taught you and you must present your argument rigorously but communicated to a general audience.
- Everyone MUST contribute (if you don’t, you get a 0)

ON TO TEACHING

- For today: I will cover optimization (in both an abstract manner and with an example)
- Make sure you review these topics carefully as you need them for the rest of your undergraduate career
- Please ask questions
- I will start with a general set up and then provide examples.

OPTIMIZATION

- Feasible vectors are $x \in \Re^n \equiv \{x = (x_i)_{i=1}^n | \forall i \in \{1,2,\cdots,n\}, x_i \in (-\infty,\infty)\}, n \in \{1,2,\cdots\} \equiv \mathbb{N}$
- Objective function (function we want to maximize): $f: \Re^n \to \Re$
- Constraint functions (functions describing the limitations faced): $\forall j \in \{1,2,\cdots,k\}, k \in \mathbb{N}, g_j: \Re^n \to \Re$

Assumptions:

- Required: functions $\left(f, (g_k)_{k=1}^k\right)$ are twice continuously differentiable
- Useful: concavity, supermodularity, monotonicity (to be explained in class)

PROBLEM AT HAND

- We want to pick an x, among those satisfying the constraints, that maximizes f: or that it solves

$$(1)\ \max_{x\in\mathfrak{R}^n} f(x)\ s.t. \forall j \in \{1,2,\cdots,k\}, g_j(x) \geq 0$$

- It is useful to first observe how to solve the problem without the constraints: i.e.

$$(2)\ \max_{x\in\mathfrak{R}^n} f(x)$$

NECESSARY AND SUFFICIENT CONDITIONS

- For a value $x \in \Re^n$ to be a local optimum, it must first satisfy the first order condition (FOC):

$$(3)\ \forall i \in \{1,2,\cdots,n\}, \frac{\partial f(x)}{\partial x_i} \equiv f_i(x) = 0.$$

- For the value to be a local minimizer, it must be that it satisfies the second order condition (SOC)
- If $n = 1$, this implies that $f''(x) \leq 0$ (or $f''(x) \geq 0$ if you want to minimize something)
- Otherwise, one requires that $Hf(x)$ be a negative semi-definite matrix. It would be a positive semi-definite if one is minimizing the objective function.
- NOTE: when f is concave, the FOC is a sufficient condition and the SOC can be ignored. Also, if a point is a local maximum and f is concave, the point is also a global maximum.
- Note: If one is minimizing the function, then one needs a convex objective function FOC are sufficient conditions for global minima.

SOME FINER POINTS

- Because you identified 1 maximum, it need not be the only one,
- When the function is strictly concave, then there exists a unique maximizer
- If the function is strictly convex, there exists a unique minimizer
- If a function is concave, then FOC works even if the function does not have a derivative at every point in its domain
- In fact (but beyond the scope of your undergraduate education), one can relax differentiability significantly.
- I will assume that functions are concave in order to disregard SOC conditions and issue of local optimality

BACK TO THE PROBLEM IN QUESTION

- Note that the FOC characterizes the objective of a function WITHOUT constraints, but who cares about this since no interesting problem has no constraints
- Lagrange pointed out that this is previous step is nonetheless useful. Here is his procedure

1. For every constraint j, define $\lambda_j \in \Re_+ = \{a \in \Re | a \geq 0\}$ as its multiplier
2. Define the function $\mathcal{L}\left(x, \left(\lambda_j\right)_{j=1}^{k}\right) = f(x) + \sum_{j=1}^{k} \lambda_j g_j(x)$
3. Solve the problem $\max_{x \in \Re^n} \mathcal{L}\left(x, \left(\lambda_j\right)_{j=1}^{k}\right)$
4. This implies FOCs of the form: $\forall i \in \{1,2, \cdots, n\}, \frac{\partial f(x^*)}{\partial x_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial g_j(x^*)}{\partial x_i} = 0$
5. Notice that at the optimum, you added a 0 to $f(x^*)$, so for every constraint j, $\lambda_j g_j(x^*) = 0$.

EXAMPLE: IN CLASS…

- Let $f(x) = u(n, b, l) = \beta\alpha \ln n + \beta(1-\alpha)\ln b + (1-\beta)\ln(1-l), \alpha, \beta \in (0,1)$
- Constraints:

1. Budget constraint: $wl + y - (n + pb) \geq 0, w, y, p \gg 0$
2. Non-negativity: $n \geq 0, b \geq 0, l \geq 0$
3. A limited number of time in a day: $l \leq 1$

EXAMPLE: 1

- Normally, the example is stated as follows:

$$(4)\ U(w, y, p) = \max_{(n,b)\in\Re^2_+, l\in[0,1]} u(n, b, l)\ \ s.t.\, n + pb \leq\ wl + y$$

- First observe that $u(.)$ is concave, so one CAN disregard the SOC
- Secondly, the function is strictly concave, so there exists a unique maximum
- Thus, the FOC and Complementary Slackness (CS) suffice to characterize the problem
- Q: But should one also decide to discard CS conditions?
- A: No, you'll see why…

EXAMPLE 2:

- Let us now set up the Lagrangian as

$$(5)\ \mathcal{L}\left(n, b, l, \lambda_0^n, \lambda_0^b, \lambda_0^l, \lambda_1^l, \lambda\right) = \beta\alpha \ln n + \beta(1-\alpha)\ln b + (1-\beta)\ln(1-l)$$

$$+\lambda_0^n n + \lambda_0^b b + \lambda_0^l l + \lambda_1^l(1-l) + \lambda[wl + y - (n + pb)]$$

- Note that $\lambda_0^n, \lambda_0^b, \lambda_0^l, \lambda \geq 0\ \left(or\ that\ \left(\lambda_0^n, \lambda_0^b, \lambda_0^l,\right) \in \Re_+^4\right)$

EXAMPLE 3

- The FOCs are then

$$(6)\ \frac{\beta\alpha}{n} + \lambda_0^n - \lambda = 0, \frac{\beta(1-\alpha)}{b} + \lambda_0^b - p\lambda = 0, \frac{-(1-\beta)}{1-l} + \lambda_0^l - \lambda_1^l + \lambda = 0$$

EXAMPLE 4:

- But we also have the CS conditions:

$$(7)\lambda_0^n n = \lambda_0^b b = \lambda_0^l l = \lambda_1^l (1-l) = \lambda[wl + y - (n + pb)] = 0.$$

EXAMPLE 5:

- Now, suppose that $n = 0$, then

$$(8)\ \frac{\beta\alpha}{0} + \lambda_0^n - \lambda = 0$$

- Notice that dividing by zero is never well defined, so one cannot define λ, λ_0^n from such condition. Hence, it must be the case that $n > 0$.
- But notice that the CS condition states that $\lambda_0^n n = 0$, so since $n > 0, \lambda_0^n = 0$.
- A similar argument shows that $b > 0, l < 1$ and hence $\lambda_0^b = \lambda_1^l = 0$.

EXAMPLE 6:

- Next, it is now clear that the FOCs become

$$(9)\ \frac{\beta\alpha}{n} = \lambda =, \frac{\beta(1-\alpha)}{pb}, \lambda_0^l + \lambda = \frac{(1-\beta)}{1-l}$$

- Notice that this implies that $\lambda > 0$, so from CS, it must be the case that

$$(10)\ pb + n = y + wl$$

EXAMPLE 7:

- Next, from the FOCs we just derived it holds that

$$(11)\ n = pb\left(\frac{\alpha}{1-\alpha}\right)$$

- If one then combine (10) and (11), it holds that

$$(12) pb + pb\left(\frac{\alpha}{1-\alpha}\right) = \frac{pb}{1-\alpha} = y + wl$$

- Or that $pb = (1-\alpha)(y+wl)$ and thus $n = \alpha(y+wl)$
- Q: But are we done?
- A: No, we need to know the value of l.

EXAMPLE 8:

- Suppose that $l > 0$, then from the FOCs it holds that

$$(13)\ \lambda = \frac{\beta\alpha}{n} = \frac{\beta}{y + wl} = \frac{1-\beta}{1-l}$$

- Or that

$$(14)\ (1-l)\beta = (1-\beta)(y + wl) \leftrightarrow l \ = \frac{\beta - (1-\beta)y}{\beta + (1-\beta)w}$$

EXAMPLE 9:

- This implies that

$$l(y,\beta) = \begin{cases} \dfrac{\beta - (1-\beta)y}{\beta + (1-\beta)w} \ if \ \dfrac{\beta}{1-\beta} > y \\ 0 \ if \ \dfrac{\beta}{1-\beta} \leq y \end{cases}$$

- It then follows that $n(y,\beta) = \alpha\big(y + wl(y,b)\big), b(y,\beta) = (1-\alpha)\big(y + wl(y,b)\big)$
- Punchline 1: If you're wealthy enough, why work!
- Punchline 2: Don't disregard the slackness conditions.

PROBABILITY THEORY

- We now move on to the second topic for today: probability theory.
- One seldom truly knows how nature operates as some features are obscured
- Q: But how does one, formally, describe how one believes that the world operate?
- A: With Probabilities!
- I will move from an example and the formalism to make things clear

EXAMPLE 1: ALICE GOES TO A STORE

- ALICE (A) COMES TO A SHOE SHOE STORE AND ASKS THE STORE EMPLOYEE, BOB (B), TO SHOW HER SOME POINTY SHOES
- BOB WANTS TO SELL ALICE THE MOST EXPENSIVE SHOES THAT ALICE IS WILLING TO BUY
- ALICE COULD BE WILLING TO SPEND UP TO $\nu \in [0,1]$, BUT BOB JUST DOES NOT KNOW ν.

SOME SET THEORY FIRST (PICTURES ARE COMING)

- A set A is just a collection of things (any collection of things can be a set
- We say that a set B is a subset of A, or that $B \subset A$, provided that for each element b in B (written as $b \in B$) b is also in A: formally $B \subset A$ if and only if (iff) $\forall b \in B, b \in A$
- An important set to note is $\emptyset$. This is the set with no elements.
- The union of sets A and B (written $A \cup B$) is the set that includes elements of either A or B: formally $A \cup B = \{c | \ c \in A \ or \ c \in B\}$
- The intersection of set A and B (written as $A \cap B$) is the set that includes only the elements that belong to BOTH A and B: formally $A \cap B = \{c | c \in A \ and \ c \in B\}$
- Assume that all the elements in question belong to some set C and $A \subset C$, then the complement of A are all of the elements of A that do not belong to A: formally, $A^C = \{c \in C | c \notin A\}$

ILLUSTRATIONS: $A \cup B$

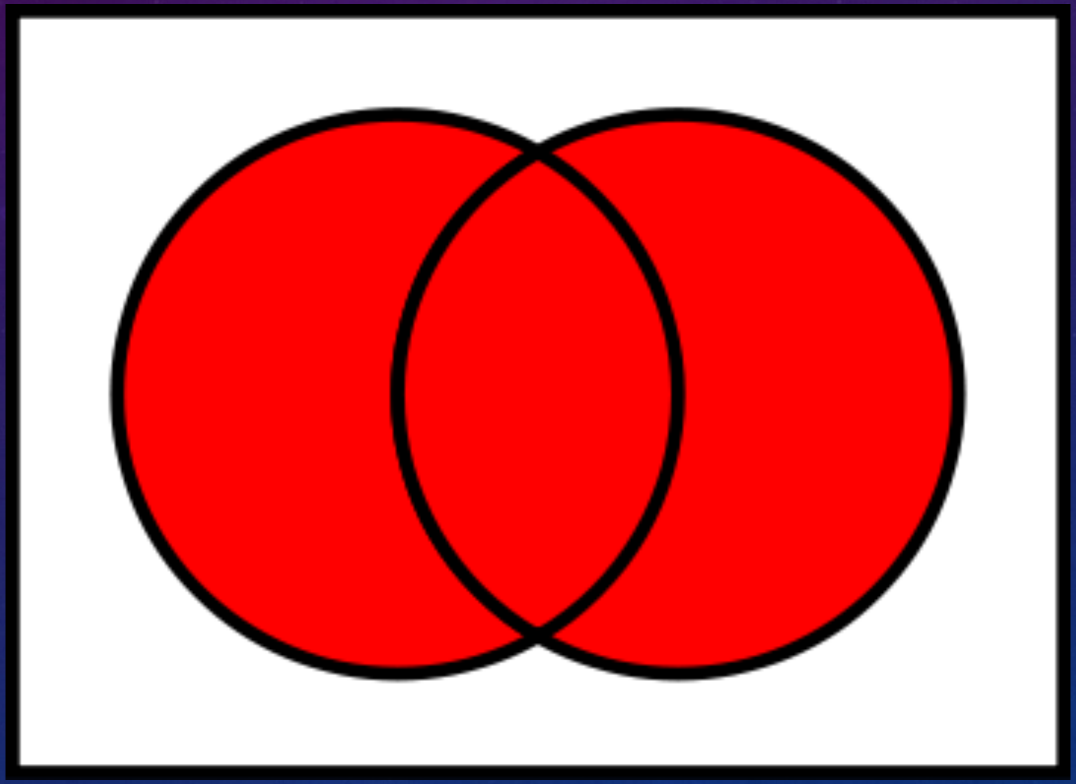


ILLUSTRATION: $A \cap B$

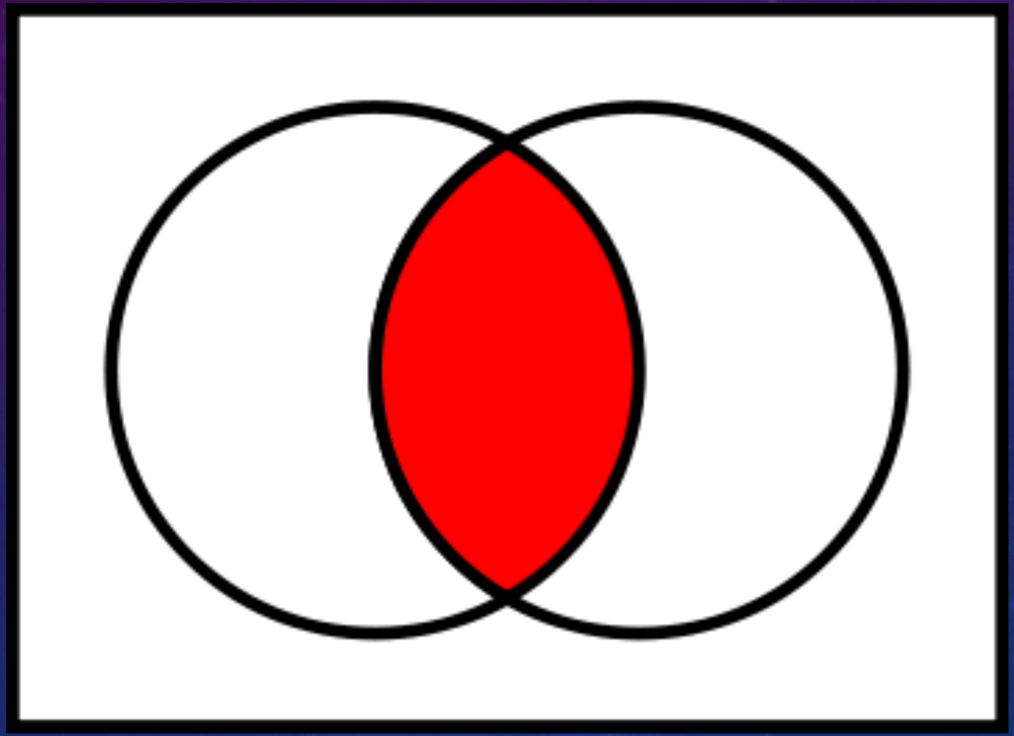
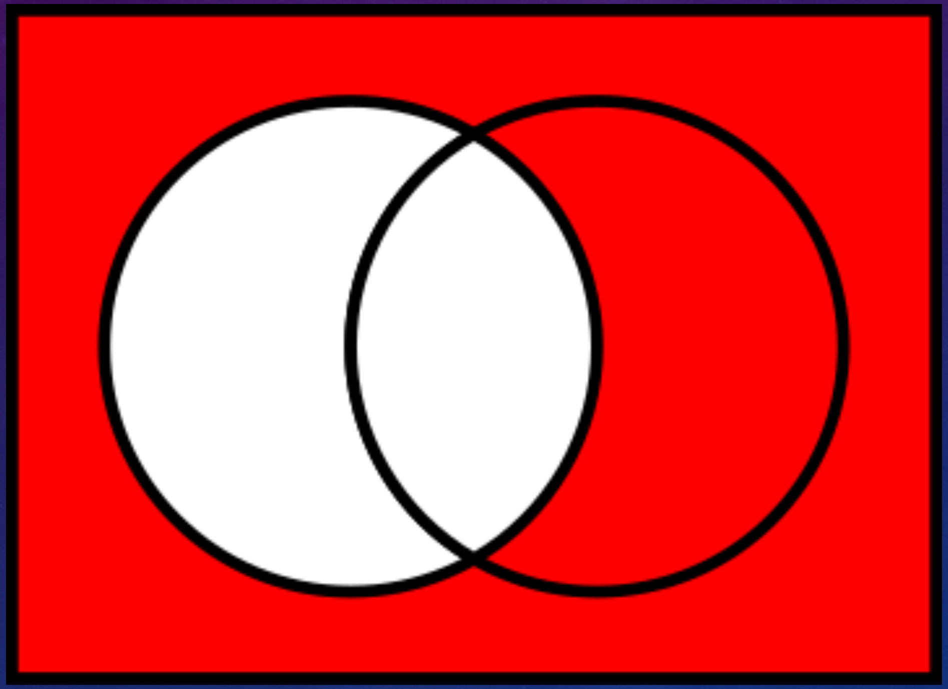
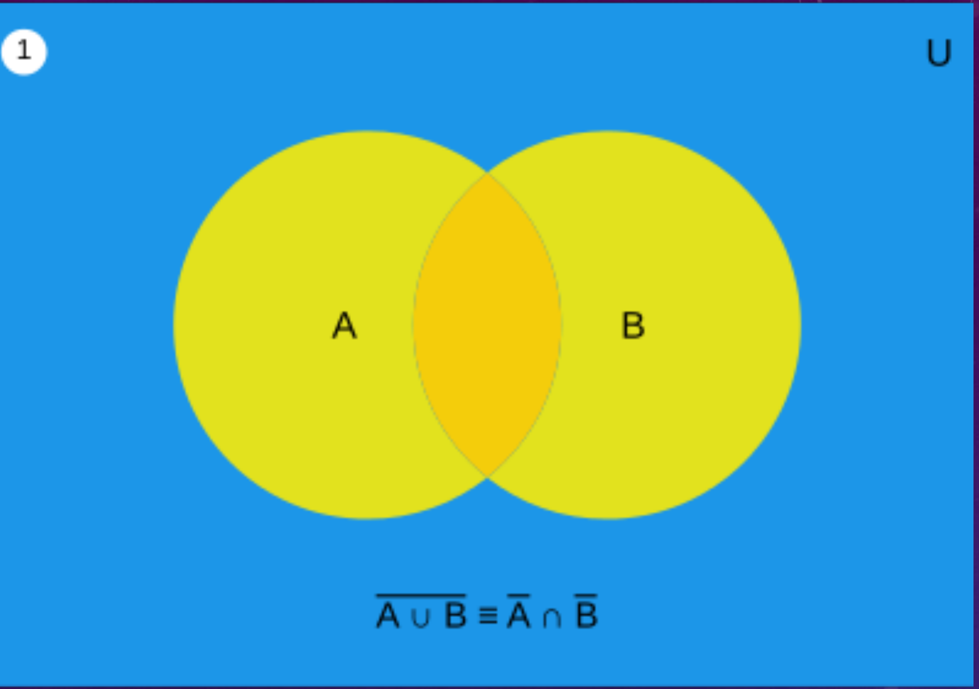
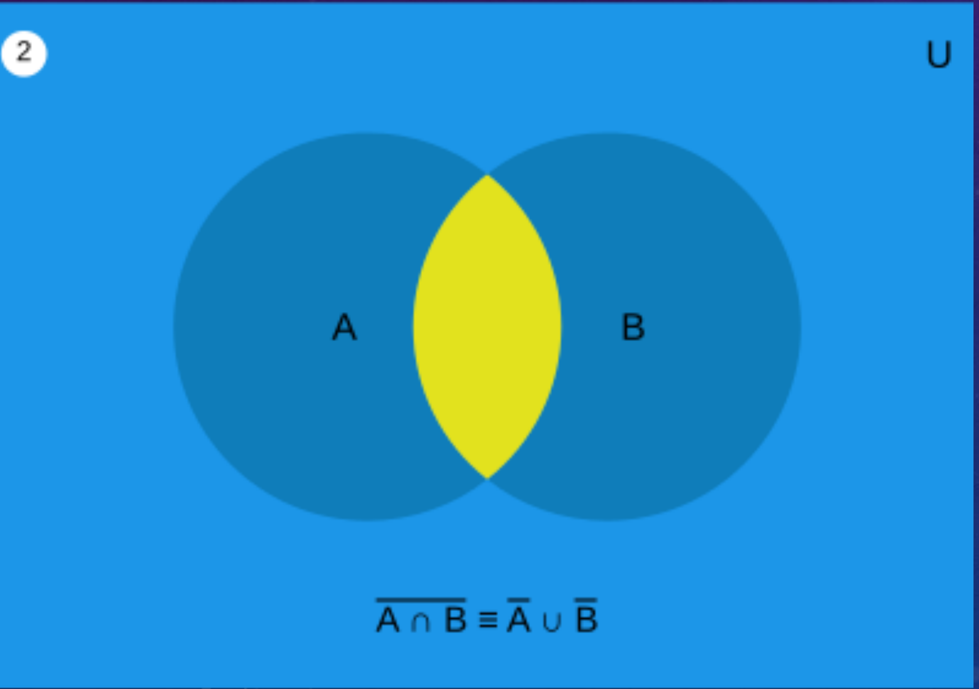


ILLUSTRATION: A^c

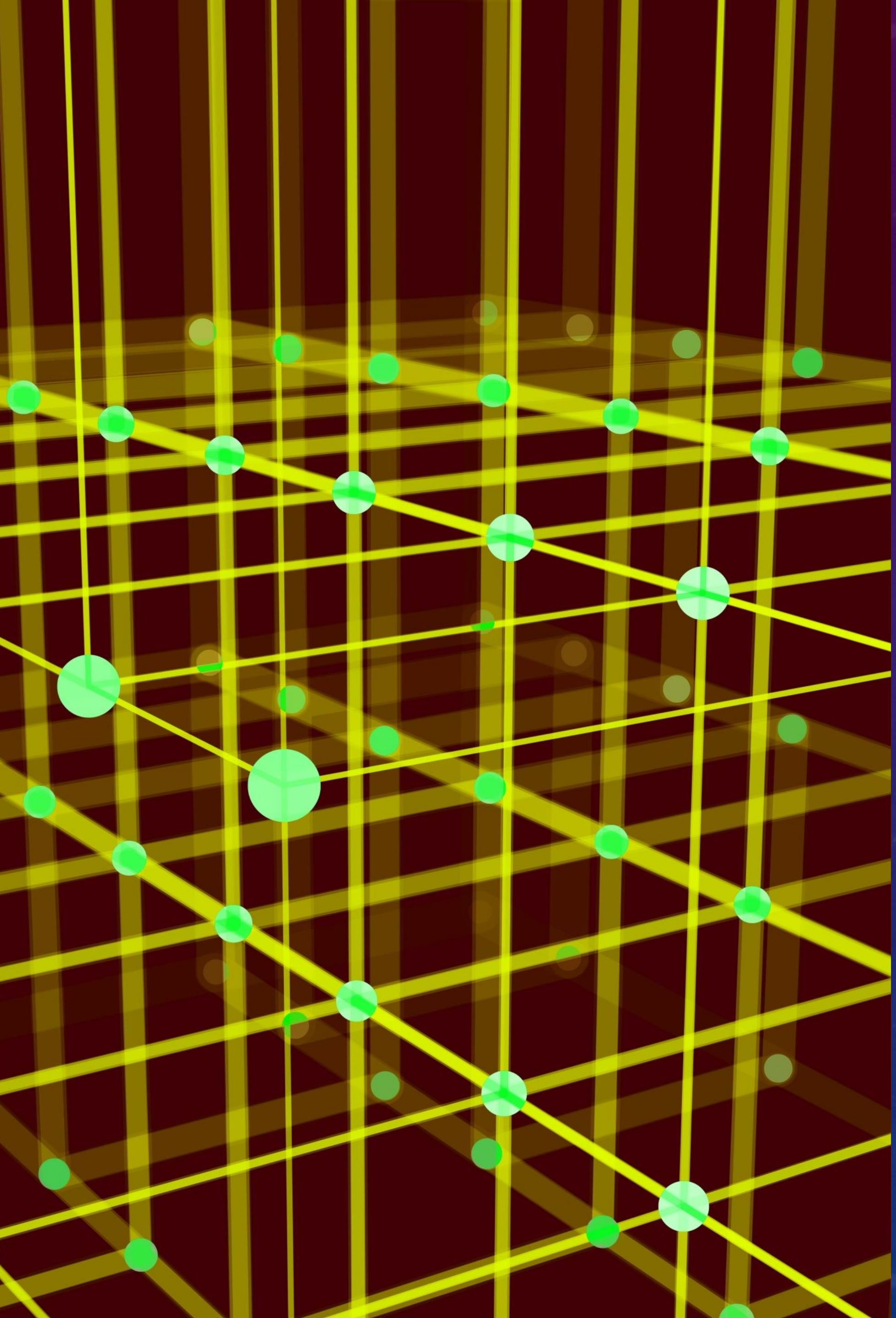






BASIC SET THEORY PROPERTIES:

1. For each set C, we can define the set of all of its subsets as $2^C = \{A | A \subset C\}$
2. Note that $\emptyset \subset C, so\ \emptyset \in 2^C$
3. De Morgan's Law: let A and B be two subsets of C, then $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$



PROBABILITY SPACE

- (How much Alice could be willing to spend) Assume that there exist a set (i.e. collection of things) of possible states of the world Ω
- (What Bob could Observe) But an individual could only observe subsets of the state of the world $\Sigma \subset 2^{\Omega}$

1. (If Bob can observe A, then he can Observe that he cannot observe A) If $A \in \Sigma$, then $A^c \in \Sigma$
2. (If Bob can observe A and B, then he can observe both things at the same time) If Bob can observe $\{A_j\}_{j=1}^{J} \subset \Sigma$, then $\cup_{j=1}^{J} A_j \in \Sigma$
3. (Bob may know precisely nothing) $\emptyset \in \Sigma$

PROBABILITIES

- Now, probabilities are given by a function $p: \Sigma \to [0,1]$ such that

1. (Odds add up to 1) $p(\Omega) = 1$
2. (Odds of completely unrelated events is the sum of each event) For every collection of events $\{A_j\}_{j=1}^{J} \subset \Sigma$ such that for each $j \neq j'$ $A_j \cap A_j' = \emptyset$, it holds that $p\left(\cup_{j=1}^{J} A_j\right) = \sum_{j=1}^{J} p\left(A_j\right)$

- A tuple (Ω, Σ, p) is called a probability space and heuristically each element means

1. Ω : the collection of things that can happen
2. Σ: the collection of things that can be observed
3. p: the odds that each thing may occur.

EXAMPLE: ALICE GOES TO THE STORE

- From Bob's point of view, he expects that $\Omega = \{v = \$50, \$150, \$1{,}500\}$
- But he only observe $\Sigma = \left\{A_1 = She\ dresses\ elegantly = \{\$150\}, A_2 = She\ doesn't = \{\$50, \$1{,}500\}\right\}$
- His beliefs are two numbers $p_1 = \Pr(A_1) \geq 0, p_2 = \Pr(A_2) \geq 0, p_1 + p_2 = 1$
- Note that if Ω has a finite number of elements, then one can describe probabilities with a collection of nonnegative numbers for each possible event: e.g. $p_{50}, p_{150}, p_{1{,}500} \geq 0, p_{50} + p_{150} + p_{1{,}500}$=1
- If Ω does not have a finite number elements this is not possible in general
- But if $\Omega \subset \Re$, then one can always define a function $F\colon \Omega \to [0,1]$ such that $\forall \omega \in \Omega, F(\omega) = \Pr(v \leq \omega)$
- If it happens to be that F is differentiable, then at least for any set A which is a countable sum of disjoint intervals belonging to Ω one can write $\Pr(A) = \int_A F'(\omega) d\omega = \int_A f(\omega) d\omega \ where\ F'(\omega) \equiv f(\omega)$

NEXT CLASS

- I conclude the discussion on probability theory and begin teaching perfect competition.